Quiz 4: Scattering Matrices and Fabry-Perots

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Lasers and Optomechanics

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General Scattering Matrix

In class we discussed a 2×2 general scattering matrix S. If we assume the scattering matrix is

1. unitary, and
2. time-reversable,

we can write the general scattering matrix as

$$\begin{array}{ccc} \boldsymbol{E}_{\text{out}} & = & S \qquad\qquad \boldsymbol{E}_{\text{in}} \\ \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}_{\text{out}} & = & e^{i\phi} \begin{bmatrix} \sqrt{1-r^2} & re^{i\delta} \\ -re^{-i\delta} & \sqrt{1-r^2} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}_{\text{in}} \end{array} \tag{1}$$

where $r \in [0, 1]$ and $\phi, \delta \in [0, 2\pi]$.

Write the parameters r, ϕ, δ needed to express the following matrices:

1. $\begin{bmatrix} t & r \\ -r & t \end{bmatrix}$ $r^2 = 1 - t^2$, $t \in [0,1]$
   $\phi = 0$, $\delta = 0$

2. $\begin{bmatrix} e^{-ikL} & 0 \\ 0 & e^{-ikL} \end{bmatrix}$ $r = 0$, $\phi = -kL$, $\delta \in [0, 2\pi]$

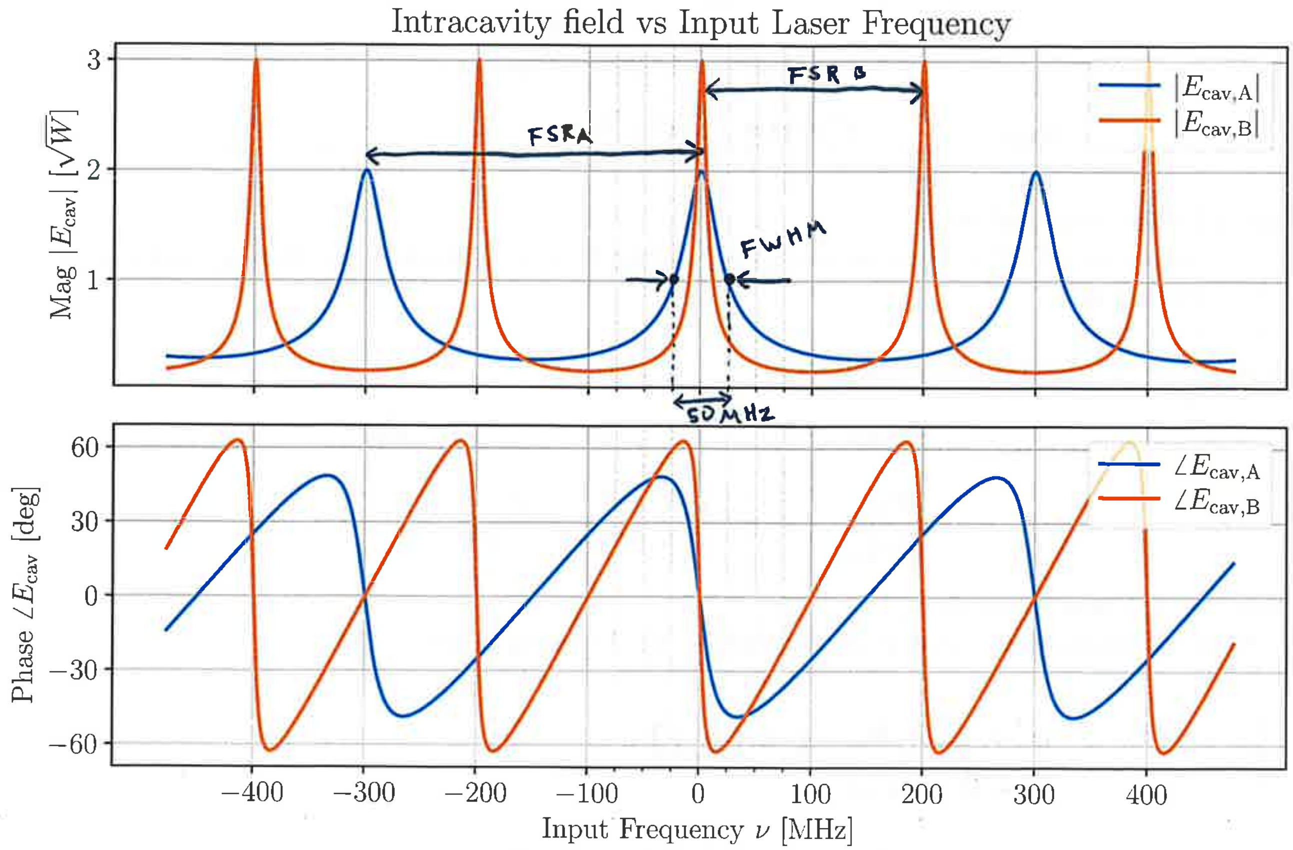
3. $\begin{bmatrix} it & r \\ r & it \end{bmatrix}$ $r^2 = 1 - t^2$, $t \in [0,1]$
   $\left.\begin{array}{l} \phi = \frac{\pi}{2}, \text{ so that } e^{i\phi} = i \\ \delta = \frac{3\pi}{2}, \text{ so that } e^{i\delta} = -i \end{array}\right\}$ $\begin{array}{l} re^{i\delta}e^{i\phi} = r(-i)(+i) = r \\ -re^{-i\delta}e^{i\phi} = -r(i)(i) = r \end{array}$

Laser Amplifier Scattering Matrix

1. Write a scattering matrix for a single-pass laser amplifier with power gain G and length L.
2. Which assumption about scattering matrices made above would no longer be valid?

1. $\begin{bmatrix} \sqrt{G}\, e^{-ikL} & 0 \\ 0 & \sqrt{G}\, e^{-ikL} \end{bmatrix}$

2. Unitarity (Power is not conserved for our seed laser)



Two Cavities Swept by One Laser

In the above figure, two cavities A and B are swept by the same laser with frequency offset ν. The intracavity fields are plotted during this sweep. For the below questions, recall the following equations,

$P_{in} = 1\,W$

$E_{in} = 1\sqrt{W}$

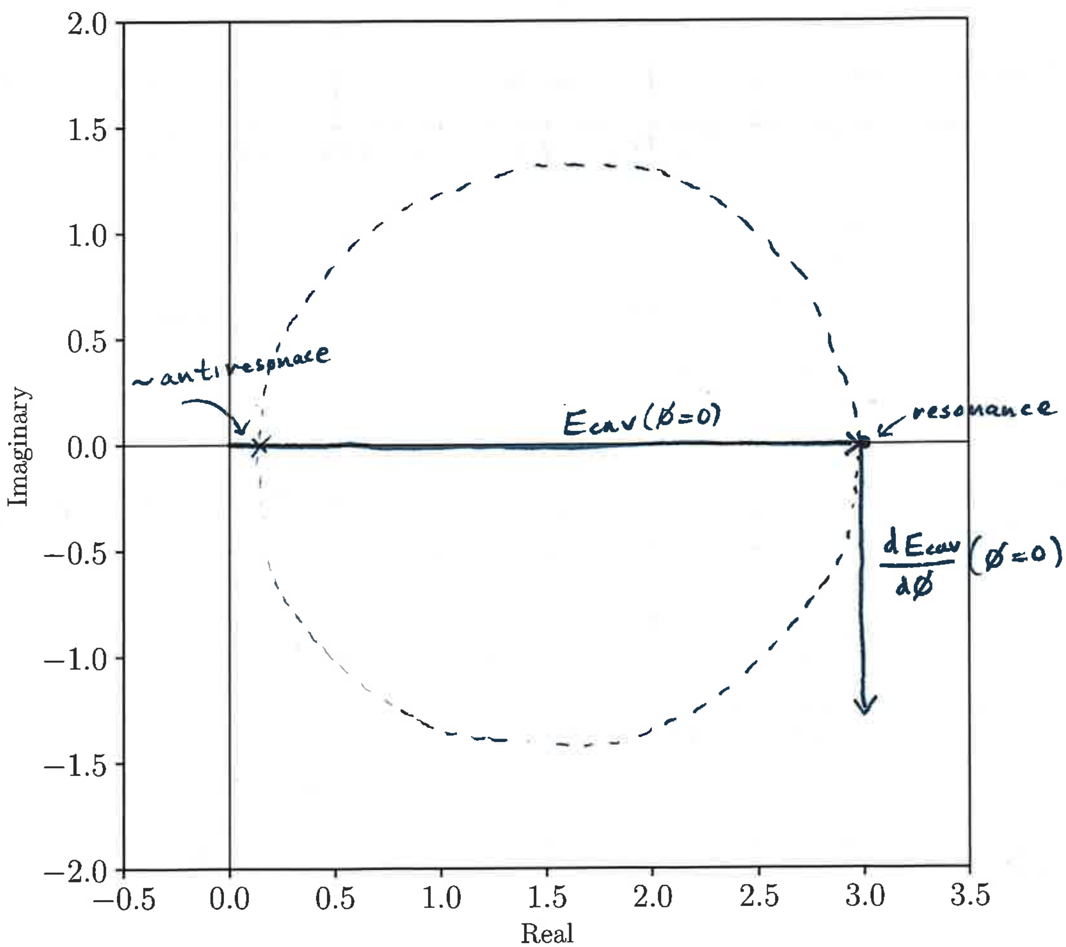
$$\frac{E_{\text{cav}}}{E_{\text{in}}} = \frac{t_1}{1 - r_1 r_2 e^{-i\phi}} \tag{2}$$

where ϕ is the round-trip phase.

1. What is the *free spectral range* FSR for the two cavities? A: 300 MHz B: 200 MHz
2. Which cavity has the longer length, A or B? B is longer, because $FSR = \frac{c}{2L}$ and $FSR_B < FSR_A$
3. What is the *finesse* $\mathcal{F}$ for cavity A? $\mathcal{F} = \frac{FSR}{FWHM} = \frac{300\text{ MHz}}{50\text{ MHz}} = 6$
4. Assuming the cavities are *critically-coupled*, what is the power transmission $T = t^2$ of the mirrors?
5. Calculate the *discriminant* of the intracavity field with respect to the round-trip phase ϕ: $\dfrac{dE_{\text{cav}}}{d\phi}$.
6. Draw the phasor for cavity B on the plot on the next page.
   Draw the vectors for E_{cav} and its discriminant $\dfrac{dE_{\text{cav}}}{d\phi}$ evaluated at resonance $\phi = 0$.
7. Is the discriminant purely a magnitude change, a phase change, or a mixture of both at resonance? What about in general?

6. Need $E_{cav} = 3$ at resonance, and some small real value at antiresonance, and a circle for the total phasor.

7. At resonance, $\phi = 0$, the change is purely phase. In general it is both.



1. $FSR_A = 300\ \text{MHz}$
   $FSR_B = 200\ \text{MHz}$ Read off of plot

2. B is longer, because $FSR = \frac{c}{2L}$ and $FSR_B < FSR_A$, so $\Rightarrow L_B > L_A$

3. Finesse $\mathcal{F}_A = \frac{FSR}{FWHM} = \frac{300\ \text{MHz}}{50\ \text{MHz}} = 6$

   read off plot

4. Power Transmission $T = t^2$ for critically coupled cavity $(r_1 = r_2 = r)$

Cavity A : $|E^{cav}_{max}| = 2\sqrt{W}$, $P_{in} = 1$ occurs at resonace $\phi = 0$

$$\frac{E_{cav}}{E_{in}}\Big|_{\phi=0} = \frac{t_1}{1 - r_1 r_2 e^{-i\phi}}\Big|_{\phi=0} = \frac{t_1}{1 - r_1 r_2}\Big|_{r_1=r_2} = \frac{t}{1-r^2}$$

$$t = \sqrt{1-r^2} \Rightarrow t^2 = 1 - r^2$$

$$\frac{E_{cav}}{E_{in}}\Big|_{\phi=0} = \frac{t}{t^2} = \frac{1}{t} = 2 \Rightarrow t = \frac{1}{2} \Rightarrow \boxed{T = \frac{1}{4}}$$

$$\cancel{\frac{P_{cav}}{P_{in}}\Big|_{\phi=0} = \frac{1}{t^2} = \frac{1}{T} = 2 \Rightarrow T = \frac{1}{2}}$$

Cavity B : $|E^{cav}_{max}| = 3\sqrt{W}$, $E_{in} = 1\sqrt{W}$

$$\left|\frac{E_{cav}}{E_{in}}\right|_{\phi=0} \Rightarrow \frac{1}{t} = 3 \Rightarrow t = \frac{1}{3} \Rightarrow \boxed{T = \frac{1}{9}}$$

5. Discriminant of E_{cav} : $\frac{dE_{cav}}{d\phi}$

$$E_{cav} = t_1 (1 - r_1 r_2 e^{-i\phi})^{-1}$$

$$\frac{dE_{cav}}{d\phi} = t_1 (-1)(1 - r_1 r_2 e^{-i\phi})^{-2}(-r_1 r_2 e^{-i\phi})(-i)$$

$$\frac{dE_{cav}}{d\phi} = \frac{-i t_1 r_1 r_2 e^{-i\phi}}{(1 - r_1 r_2 e^{-i\phi})^2}$$

Critically coupled

Cavity B Values

Evaluate at resonance

$$\frac{dE_{cav}}{d\phi}\Big|_{\phi=0} = \frac{-i t_1 r_1 r_2}{(1 - r_1 r_2)^2} = -i \frac{r_1 r_2}{t_1} E^2_{cav(\phi=0)} = -i \frac{r^2}{t} E^2_{cav,0} = -i \frac{(8/\cancel{9})}{(1/3)}(\cancel{9}) \boxed{= -i24}$$