Quiz 2: Waves and Interference

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Lasers and Optomechanics

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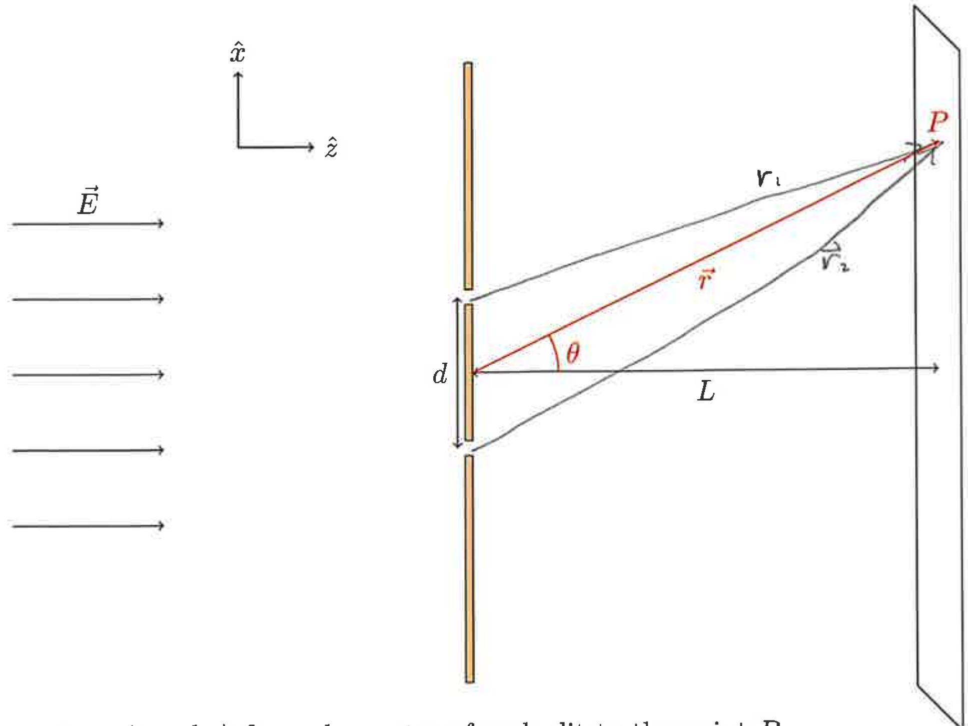
Plane Wave incident on a Double Slit

Suppose we have two narrow slits cut a distance d apart into a opaque surface in the xy plane. Incident on the surface is a plane wave electric field $\vec{E} = E_0 \exp(ikz)\hat{x}$ propogating in the $+\hat{z}$ direction. Finally, we have another surface in the xy plane a distance L away where we would like to know the intensity $I(\theta)$, where θ is the angle from the middle of the slits to a point on the surface at L. We will assume $d \ll L$.



1. Draw the vectors $\vec{r}_1$ and $\vec{r}_2$ from the center of each slit to the point P.

2. Write expressions for $\vec{r}_1$ and $\vec{r}_2$, and $|r_1|$ and $|r_2|$ in terms of L, d and θ.
Hint 1: Find an expression for $\vec{r}$ first.
Hint 2: It may be helpful that $1 + \tan^2\theta = \dfrac{1}{\cos^2\theta}$

3. Write an first-order approximation for $|r_2|$ - $|r_1|$.
Hint: Assume $d \ll L$, and use the binomial approximation $(1+\epsilon)^n \approx 1 + n\epsilon$ for $\epsilon \ll 1$.

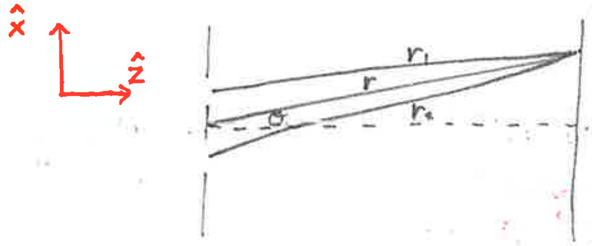
4. Assuming that the output from each slit is a spherical wave $E_i = \dfrac{A}{|\vec{r}_i|} e^{i\vec{k}\cdot\vec{r}_i}$, write a general expression for the intensity I in terms of r_1 and r_2.

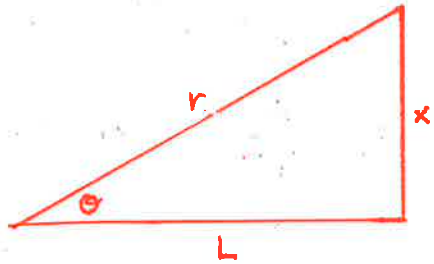
5. Simplify your expression for the intensity I in terms of θ, removing all references to r_1, r_2.
Hint: For simplicity, expand the front fraction $\dfrac{A}{r_i}$ to zeroth order, so that $\dfrac{A}{|\vec{r}_1|} = \dfrac{A}{|\vec{r}_2|} = \dfrac{A}{|\vec{r}|}$.

6. At what θ would $I(\theta)$ first equal zero, if at all?

7. Suppose now the incident plane wave is at an angle ϕ, such that the waves struck slit 1 first. How would your new intensity expression $I(\theta, \phi)$ change compared to $I(\theta)$?

1.





$x = L\tan\theta$

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2. $\vec{r}_1 = \vec{r} - \frac{\vec{d}}{2}$ $\qquad \vec{r} = L\hat{z} + L\tan\theta\,\hat{x}$ $\qquad \vec{r}_1 = L\hat{z} + \left(L\tan\theta - \frac{d}{2}\right)\hat{x}$

$\vec{r}_2 = \vec{r} + \frac{\vec{d}}{2}$ $\qquad \vec{d} = d\hat{x}$ $\qquad \vec{r}_2 = L\hat{z} + \left(L\tan\theta + \frac{d}{2}\right)\hat{x}$

$$|r_1| = \sqrt{L^2 + \left(L\tan\theta - \frac{d}{2}\right)^2}$$

$$= \left(L^2 + L^2\tan^2\theta + \frac{d^2}{4} - dL\tan\theta\right)^{\frac{1}{2}}$$

$$= \left(L^2(1+\tan^2\theta) + \frac{d^2}{4} - dL\tan\theta\right)^{\frac{1}{2}}$$

$$|r_1| = \left(\frac{L^2}{\cos^2\theta} + \frac{d^2}{4} - dL\tan\theta\right)^{\frac{1}{2}}$$

$$|r_2| = \sqrt{L^2 + \left(L\tan\theta + \frac{d}{2}\right)^2}$$

$$|r_2| = \left(\frac{L^2}{\cos^2\theta} + \frac{d^2}{4} + dL\tan\theta\right)^{\frac{1}{2}}$$

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3. $|r_2| - |r_1| = ?$

$$|r_1| = \left(\frac{L^2}{\cos^2\theta} + \overset{0}{\cancel{\frac{d^2}{4}}} - dL\tan\theta\right)^{\frac{1}{2}}$$

$$= \frac{L}{\cos\theta}\left(1 - \frac{d}{L}\sin\theta\cos\theta\right)^{\frac{1}{2}}$$

$$|r_1| \simeq \frac{L}{\cos\theta}\left[1 - \frac{1}{2}\frac{d}{L}\sin\theta\cos\theta\right] = \frac{L}{\cos\theta} - \frac{d}{2}\sin\theta$$

$$|r_2| \simeq \frac{L}{\cos\theta}\left[1 + \frac{1}{2}\frac{d}{L}\sin\theta\cos\theta\right] = \frac{L}{\cos\theta} + \frac{d}{2}\sin\theta$$

$$|r_2| - |r_1| = d\sin\theta$$

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4. $$I(\vec{r}_1, \vec{r}_2) = \left|\frac{A}{|r_1|}e^{i\vec{k}\cdot\vec{r}_1} + \frac{A}{|r_2|}e^{i\vec{k}\cdot\vec{r}_2}\right|^2$$

5. $$I(\theta) = \frac{A^2}{|r|^2}\left| e^{i\vec{k}\cdot\vec{r}_1} + e^{+i\vec{k}\cdot\vec{r}_2}\right|^2$$

$$= \frac{A^2}{|r|^2}\left| e^{ikr_1}\left(1 + e^{i\underbrace{k(r_2-r_1)}}\right)\right|^2 \quad , \quad \left|e^{ikr_1}\right|^2 = 1$$

$$= \frac{A^2}{|r|^2}\left| e^{ikr_1}\left(1 + e^{\overbrace{ikd\sin\theta}}\right)\right|^2$$

$$= \frac{A^2}{|r|^2}\left(\underbrace{e^{ikr_1}}_{1}\left(1 + e^{ikd\sin\theta}\right)\right)\overbrace{\left(\underbrace{e^{-ikr_1}}_{1}\left(1 + e^{-ikd\sin\theta}\right)\right)}^{\text{Complex Conjugate}}$$

$$= \frac{A^2}{|r|^2}\left[1 + 1 + e^{ikd\sin\theta} + e^{-ikd\sin\theta}\right]$$

$$= \frac{2A^2}{|r|^2}\left[1 + \cos(kd\sin\theta)\right] \quad , \quad |r| = \frac{L}{\cos\theta}$$

$$I(\theta) = \frac{2A^2\cos^2\theta}{L^2}\left[1 + \cos(kd\sin\theta)\right]$$

6. $I(\theta) = 0$ when $1 + \cos(kd\sin\theta) = 0$, or $\cos(kd\sin\theta) = -1$

$$\Rightarrow kd\sin\theta = \pi(2n+1)$$

First zero : $n = 0$

$$\boxed{\sin\theta = \frac{\pi}{kd}} \qquad k = \frac{2\pi}{\lambda} \Rightarrow \sin\theta = \frac{\lambda}{2d}(2n+1)$$

7. $I(\theta,\phi) = |E_1 + E_2(\phi)|$, so the arrival times of the waves are now staggered

$$= \frac{A^2}{|r|^2}\left| e^{i\vec{k}\cdot\vec{r}_1 + \frac{\phi}{2}} + e^{i\vec{k}\cdot\vec{r} - \frac{\phi}{2}}\right|$$

The peaks and throughs will shift

Full answer : The spherical waves are no longer in-phase

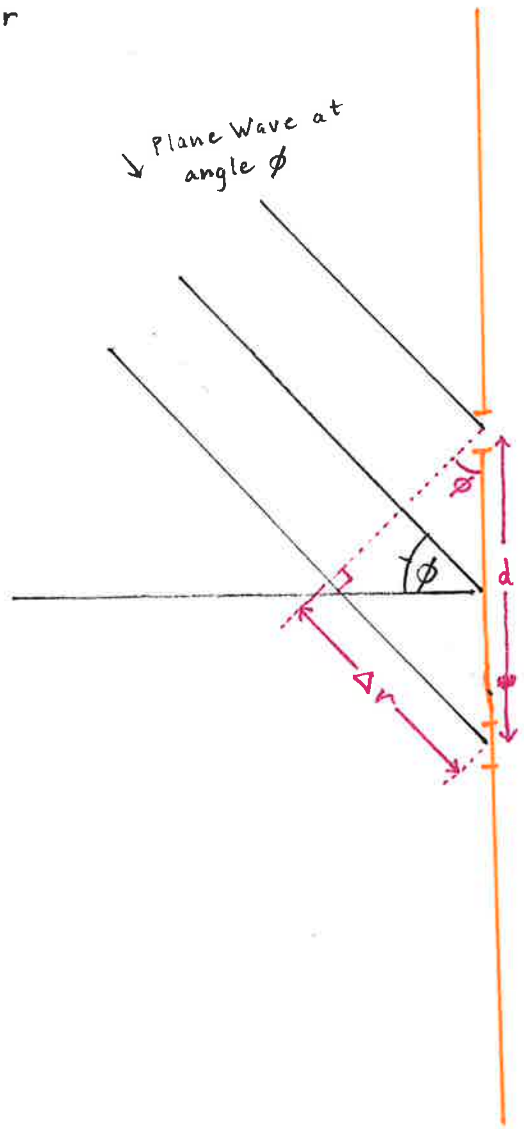
$$I(\theta,\phi) = \frac{A^2}{|r|^2}\left| e^{ikr_1 + \frac{\phi}{2}}\left(1 + e^{ik(r_1-r_2)+\phi}\right)\right|^2$$

$$= \frac{2A^2}{|r|^2}\left[1 + \cos(kd\sin\theta + \phi)\right]$$

Sort of...

Full Solution for (7)

$E_0 e^{ikr}$



If the angle of incidence of the plane wave really is ϕ away from normal incidence, then the wave emitted from the top slit will have an earlier phase and the bottom slit will have a later phase.

Quantitatively, the extra phase accrued we'll call φ, and will be equal to

$$\varphi = k\Delta r$$

$$\Delta r = d\sin\phi$$

$$\Rightarrow \varphi = kd\sin\phi$$

Then a more correct expression for (7) is

$$I(\theta,\phi) = \frac{A^2}{|r|^2}\left| e^{i\vec{k}\cdot\vec{r}_1 - \frac{\varphi}{2}} + e^{i\vec{k}\cdot\vec{r}_2 + \frac{\varphi}{2}}\right|^2$$

$$= \frac{2A^2}{|r|^2}\left[1 + \cos\left(kd(\sin\theta + \sin\phi)\right)\right]$$