**Quiz 1: Complex Numbers Review**

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Lasers and Optomechanics

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## Question 1: Complex Geometry

√ 1. Plot and label the point $z = -2 + 2i$.

√ 2. Write $z$ in terms of its polar coordinates $r$ and $\theta$.

√ 3. Plot and label its complex conjugate $z^*$. $= -2-2i$

√ 4. Plot and label the sum $z + z^* = -4$

√ 5. Plot and label the difference $z - z^* = 4i$

√ 6. Plot and label the product $\frac{z^2}{2} = \frac{1}{2} r^2 e^{i2\theta} = 4e^{\frac{3\pi}{2}} = -4i$

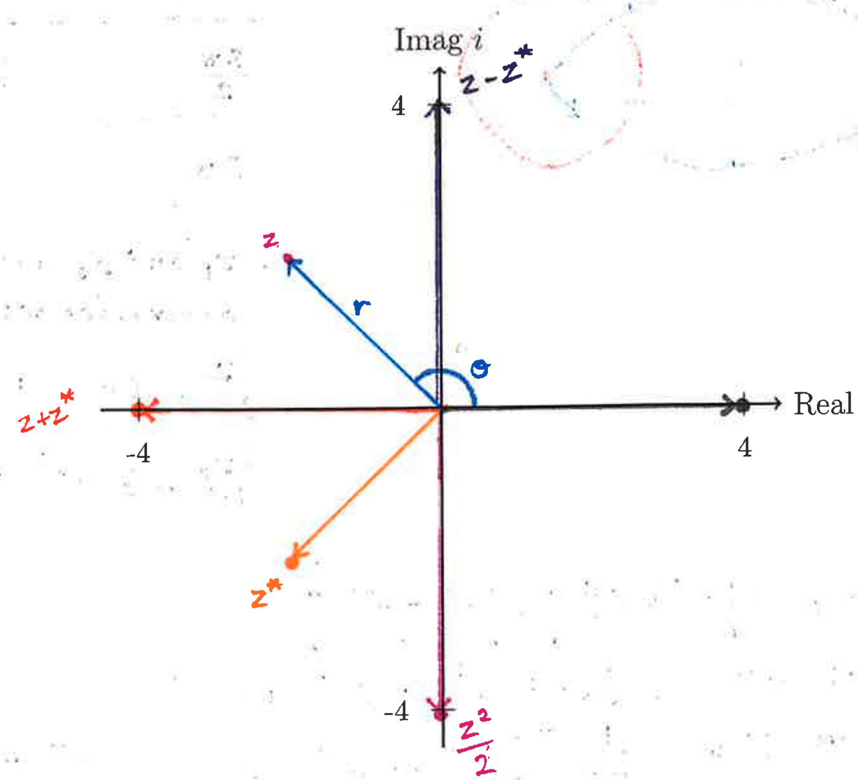
√ 7. Plot and label the product $\frac{|z|^2}{2} = \frac{1}{2} r^2 = 4$

$z = a + ib = re^{i\theta}$

$r = \sqrt{a^2+b^2} = \sqrt{(-2)^2 + 2^2} = \sqrt{8}$

$\theta = \arctan2(-2, 2) = \frac{3\pi}{4}$

Without a calculator, need to do this geometrically: $\theta = 135°$ from the plot of $z$, and using $\pi$ rads $= 180°$ we can get back to radians.



$z = a + ib = re^{i\theta}$

**Question 2: Complex Functions**

Suppose we have a complex functions $z_1$, $z_2$, and $z_3$ defined as a function of $\phi \in [0, 2\pi]$:

$$\checkmark \; z_1(\phi) = 2 + e^{i\phi} \tag{1}$$

$$\checkmark \; z_2(\phi) = \frac{1}{2}(e^{i\phi} + 3e^{-i\phi}) \qquad z_2(\phi=\tfrac{\pi}{2}) = \tfrac{1}{2}(i - 3i) = -i \tag{2}$$

$$\checkmark \; z_3(\phi) = \frac{3\sqrt{2}ie^{i\frac{\pi}{4}}}{2}(e^{i\phi} + e^{-i\phi}) = 3\sqrt{2}\, i e^{i\frac{\pi}{4}} \cos\phi \tag{3}$$

At $\phi = 0$, $z_3(0) = 3\sqrt{2}\, i e^{i\frac{\pi}{4}}$

Remember that $i$ is just a 90° rotation: $i = e^{i\frac{\pi}{2}}$

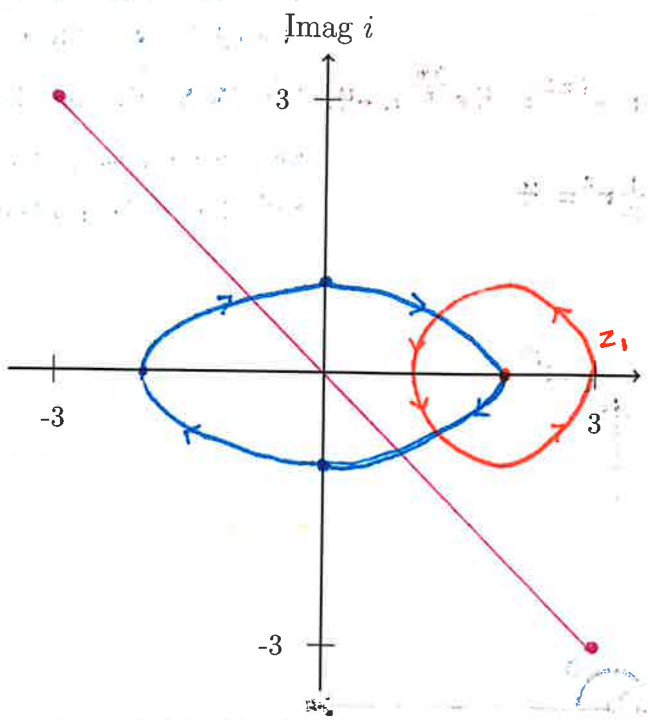
$\Rightarrow z_3(0) = \underbrace{3\sqrt{2}}_{r}\, \underbrace{e^{i\frac{3\pi}{4}}}_{\arg} = a + ib$

$a = 3\sqrt{2}\cos(\frac{3\pi}{4}$

$a = -3$

$b = 3$

1. Plot and label $z_1$, $z_2$, and $z_3$ over $\phi \in [0, 2\pi]$.
2. What shapes do $z_1$, $z_2$, and $z_3$ form? $\boxed{z_1\text{: Circle},\ z_2\text{: Ellipse},\ z_3\text{: Line}}$
3. Which way are $z_1$ and $z_2$ rotating as $\phi$ increases? $\boxed{z_1\text{: CCW},\ z_2\text{: CW}}$
4. How can you know which way $z_1$ and $z_2$ are rotating? Justify it with a calculation.




| $\phi$ | $z_1$ | $z_2$ | $z_3$ |
|---|---|---|---|
| 0 | 3 | 2 | $-3+3i$ |
| $\frac{\pi}{2}$ | $2+i$ | $-i$ | 0 |
| $\pi$ | 1 | $-2$ | $3-3i$ |
| $\frac{3\pi}{2}$ | $2-i$ | $i$ | 0 |
| $2\pi$ | | | |

4. $z_1$ and $z_2$ rotation: Take derivative with respect to $\phi$:

$$\frac{dz_1}{d\phi} = ie^{i\phi} = \underline{i \text{ at } \phi = 0}$$

$$\frac{dz_2}{d\phi} = \frac{1}{2}(ie^{i\phi} - 3ie^{-i\phi}) = \underline{-i \text{ at } \phi=0}$$

**Question 3: Fourier Transform**

Calculate the Fourier Transform of $x(t) = 2\cos(t) + \sin(t)$. Recall that $X(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt$

$$x(t) = 2\cos(t) + \sin(t) = 2\left[\frac{e^{it} + e^{-it}}{2}\right] + \left[\frac{e^{it} - e^{-it}}{2i}\right]$$

Collect by e-terms: $x(t) = \frac{1}{2}(2-i)e^{it} + \frac{1}{2}(2+i)e^{-it}$

Set up Fourier Transform: $X(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{2}(2-i)e^{it} + \frac{1}{2}(2+i)e^{-it}\right]e^{-i\omega t}dt$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{2}(2-i)\underbrace{\int_{-\infty}^{\infty} e^{i(1-\omega)t}dt}_{2\pi\delta(\omega-1)} + \frac{1}{2}(2+i)\underbrace{\int_{-\infty}^{\infty} e^{-i(1+\omega)t}dt}_{2\pi\delta(1+\omega)}\right]$$



$$X(\omega) = \sqrt{\frac{\pi}{2}}\left[(2-i)\delta(\omega-1) + (2+i)\delta(\omega+1)\right]$$