Fabry Perot Dark Offset Tutorial Solutions

Part 1: Transmitted Power

Define the transmitted field function (assuming Ein = 1 $\sqrt{W}$)

$$\text{Etrans}\left[\phi\_,\ \text{t1}\_ : 0.3,\ \text{t2}\_ : 0.3,\ \text{r1}\_ : \sqrt{1-0.3^2},\ \text{r2}\_ : \sqrt{1-0.3^2}\right] := \frac{\text{t1 t2 Exp}[-\text{I}\ \phi]}{1-\text{r1 r2 Exp}[-\text{I}\ 2\ \phi]};$$

1. Derive an expression for the transmitted power $P_{\text{trans}}(\phi)$

In[92]:=
```
Etrans0 = Etrans[ϕ, t1, t2, r1, r2];
Ptrans = Simplify[ComplexExpand[Etrans0 Conjugate[Etrans0]]]
```

Out[93]=

$$\frac{\text{t1}^2\ \text{t2}^2}{1+\text{r1}^2\ \text{r2}^2-2\ \text{r1}\ \text{r2}\ \text{Cos}[2\ \phi]}$$

2. What is the maximum transmitted power at resonance $P_{\text{trans}}(0)$?

In[96]:=
```
Ptrans /. {ϕ → 0}
```

Out[96]=

$$\frac{\text{t1}^2\ \text{t2}^2}{1-2\ \text{r1}\ \text{r2}+\text{r1}^2\ \text{r2}^2}$$

3. Starting at resonance, what happens to P_{trans} when the cavity gets longer?

The power goes down like ϕ^2.
Looking at the equation for P_{trans}, the cosine in the denominator ensures that the offset must be significant before we start to see change.
Quantitatively, we could take the derivative of P_{trans} with respect to ϕ, then evaluate at $\phi = 0$

In[98]:=
```
dPtransdϕ = Simplify[D[Ptrans, ϕ]]
```

Out[98]=

$$-\frac{4\ \text{r1}\ \text{r2}\ \text{t1}^2\ \text{t2}^2\ \text{Sin}[2\ \phi]}{\left(1+\text{r1}^2\ \text{r2}^2-2\ \text{r1}\ \text{r2}\ \text{Cos}[2\ \phi]\right)^2}$$

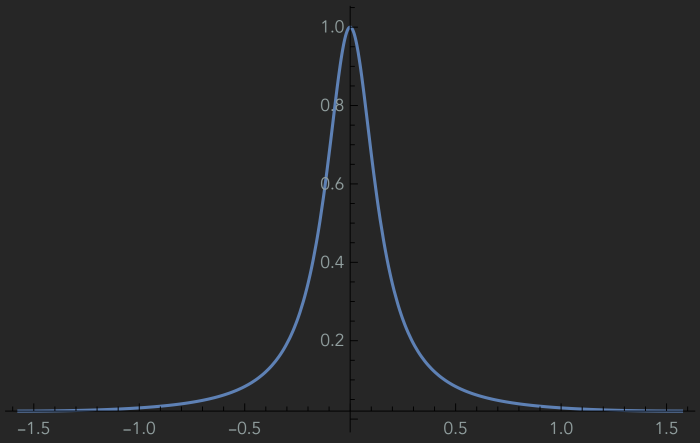
```
In[99]:= dPtransdϕ /. {ϕ → 0}
Out[99]=
         0
```

4. Starting at resonance, what happens to P_{trans} when the cavity gets shorter?

Again, the power goes down like ϕ^2.

This is equivalent because the P_{trans} function is symmetric. If one considers $\phi \to -\phi$ in the P_{trans} function, we get the same value as before because Cos[-ϕ] = Cos[ϕ]

```
In[110]:=
         Plot[Ptrans //. {r2 → r1, t2 → t1, r1 → √(1 - t1²), t1 → 0.5},
          {ϕ, -π/2, π/2}, PlotRange → Full]
Out[110]=
```



5. Would it be possible to use the transmitted power to restore resonance in case of some disturbance ΔL?

Perhaps.

You would have to move in a direction and monitor whether the transmitted power went up or down.

If transmitted power went down, then you'd need to switch directions.

It would be hard to know when you achieved perfect resonance as well, as you'd tend to go past it and need to reverse directions.

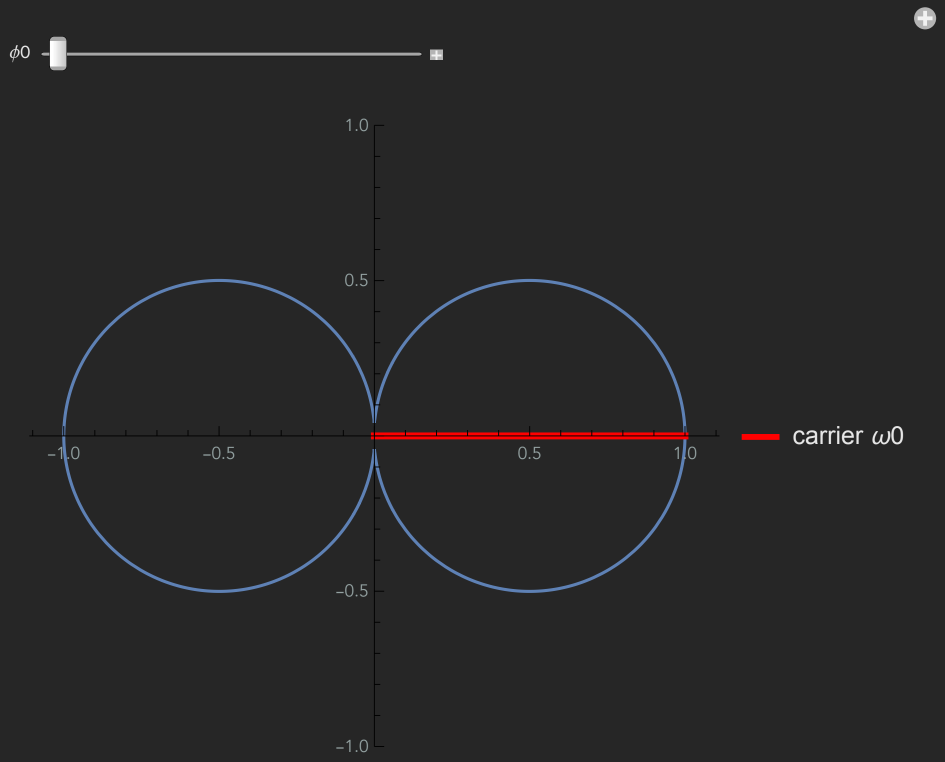
Extra Credit: Transmitted Field Phasor Plots

We get two "lobes" for even and odd resonances, thanks to the Exp[-I ϕ] in the numerator.

In[111]:=

```
Manipulate[Show[{ParametricPlot[{Re[Etrans[ϕ]], Im[Etrans[ϕ]]},
    {ϕ, 0, 2 π}, PlotRange → {-1, 1}, PlotPoints → 200, MaxRecursion → 10],
   ListLinePlot[{{0, 0}, {Re[Etrans[ϕ0]], Im[Etrans[ϕ0]]}},
    PlotStyle → {Thickness[0.01], Red}, PlotLegends → {"carrier ω0"}]
  }],
 {ϕ0, 0, 2 π}, Paneled → False]
```

Out[111]=



Part B: Reflected Power

Define the REFL electric field

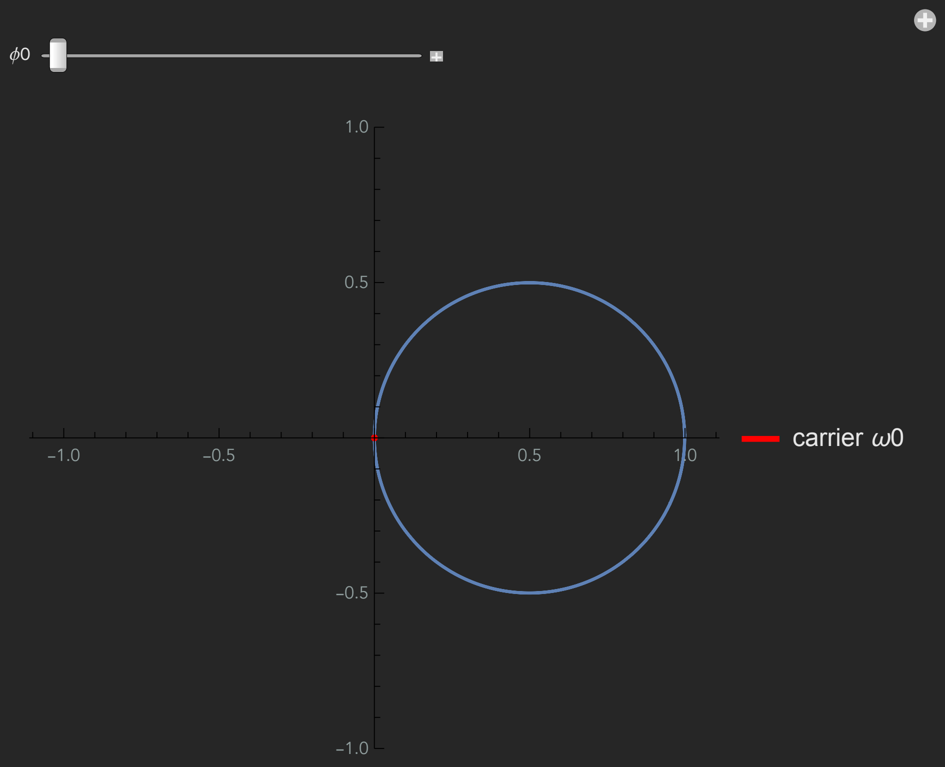
In[35]:=

$$\mathtt{Erefl}\left[\phi\_,\ \mathtt{t1\_}:0.3,\ \mathtt{t2\_}:0.3,\ \mathtt{r1\_}:\sqrt{1-0.3^{2}},\ \mathtt{r2\_}:\sqrt{1-0.3^{2}}\right] := \frac{\mathtt{r1} - \mathtt{r2}\ \mathtt{Exp}[-\mathtt{I}\ 2\ \phi]}{1 - \mathtt{r1}\ \mathtt{r2}\ \mathtt{Exp}[-\mathtt{I}\ 2\ \phi]};$$

In[36]:=

```
Manipulate[Show[{ParametricPlot[{Re[Erefl[ϕ]], Im[Erefl[ϕ]]},
    {ϕ, 0, 2 π}, PlotRange → {-1, 1}, PlotPoints → 200, MaxRecursion → 10],
   ListLinePlot[{{0, 0}, {Re[Erefl[ϕ0]], Im[Erefl[ϕ0]]}},
    PlotStyle → {Thickness[0.01], Red}, PlotLegends → {"carrier ω0"}]
  }],
 {ϕ0, 0, 2 π}, Paneled → False]
```

Out[36]=



1. Derive an expression for the reflected power $P_{\mathrm{refl}}(\phi)$. What is the reflected power at resonance?

In[39]:=

```
Erefl0 = Erefl[ϕ, t1, t2, r1, r2]
```

Out[39]=

$$\frac{r1 - e^{-2 i \phi}\, r2}{1 - e^{-2 i \phi}\, r1\, r2}$$

In[44]:=

```
Prefl = Simplify[ComplexExpand[Erefl0 Conjugate[Erefl0]]]
```

Out[44]=

$$\frac{r1^{2} + r2^{2} - 2\, r1\, r2\, \mathrm{Cos}[2\,\phi]}{1 + r1^{2}\, r2^{2} - 2\, r1\, r2\, \mathrm{Cos}[2\,\phi]}$$

```
In[46]:= Simplify[Prefl /. {ϕ → 0}]
```

Out[46]=

$$\frac{(r1 - r2)^2}{(-1 + r1\ r2)^2}$$

2. Set $r_2 = r_1 = r$. Does $P_{\mathrm{refl}}(\phi)$ simplify at all? What value do you get at resonance?

We get zero reflected power for a critically-coupled cavity at resonance.

```
In[102]:=
Prefl2 = Simplify[Prefl /. {r2 → r, r1 → r}]
```

Out[102]=

$$\frac{4\ r^2\ \mathrm{Sin}[\phi]^2}{1 + r^4 - 2\ r^2\ \mathrm{Cos}[2\ \phi]}$$

```
In[101]:=
Prefl2 /. {ϕ → 0}
```

Out[101]=

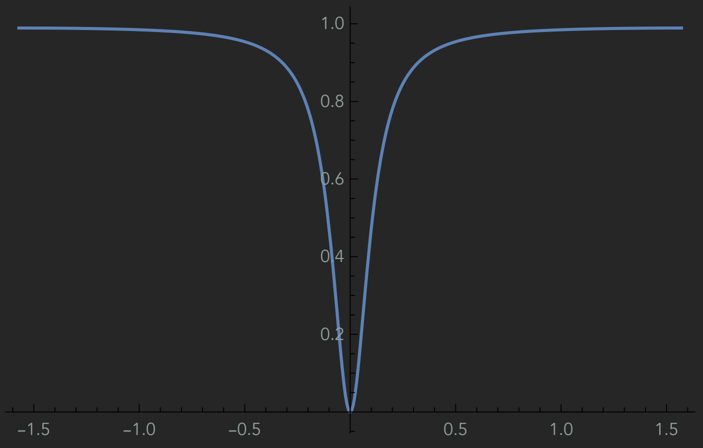
```
0
```

3. Starting at resonance, what happens to the reflected power when the cavity gets longer? Shorter?

Here, the reflected power goes up like ϕ^2 for both the cavity getting longer and shorter, which can again be seen from the symmetric $P_{\mathrm{refl}}(\phi)$ function.

```
In[106]:=
Plot[Prefl2 /. {r → 0.9}, {ϕ, -π/2, π/2}, PlotRange → Full]
```

Out[106]=



4. Can you use reflected power to restore resonance?

You would run into the same problems as for the transmitted power.

Part C: The Reflected Field

1. Derive an expression for the change in reflected field with respect to phase ϕ: $\frac{dE_{\text{refl}}}{d\phi}(\phi)$.

```
In[47]:= D[Erefl0, ϕ]
```

Out[47]=

$$-\frac{2\,i\,e^{-2\,i\,\phi}\,r1\,r2\,\left(r1-e^{-2\,i\,\phi}\,r2\right)}{\left(1-e^{-2\,i\,\phi}\,r1\,r2\right)^{2}}+\frac{2\,i\,e^{-2\,i\,\phi}\,r2}{1-e^{-2\,i\,\phi}\,r1\,r2}$$

```
In[49]:= dErefl0dϕ = Simplify[D[Erefl0, ϕ]]
```

Out[49]=

$$-\frac{2\,i\,e^{2\,i\,\phi}\,\left(-1+r1^{2}\right)\,r2}{\left(e^{2\,i\,\phi}-r1\,r2\right)^{2}}$$

2. Set $r_2 = r_1 = r$. What is $E_{\text{refl}}(0)$? What is $\frac{dE_{\text{refl}}}{d\phi}(0)$?

```
In[114]:=
Simplify[Erefl0 /. {r2 → r, r1 → r, ϕ → 0}]
```

Out[114]=

$$0$$

```
In[112]:=
Simplify[dErefl0dϕ /. {r2 → r, r1 → r}]
```

Out[112]=

$$-\frac{2\,i\,e^{2\,i\,\phi}\,r\,\left(-1+r^{2}\right)}{\left(e^{2\,i\,\phi}-r^{2}\right)^{2}}$$

```
In[115]:=
Simplify[dErefl0dϕ /. {r2 → r, r1 → r, ϕ → 0}]
```

Out[115]=

$$-\frac{2\,i\,r}{-1+r^{2}}$$

Again, $E_{\text{refl}}(0) = 0$, but $\frac{dE_{\text{refl}}}{d\phi}(0) = \frac{2 i r}{1-r^2}$.

So we have *no* reflected field returning in reflection exactly at resonance, but the field is moving rapidly in the imaginary direction at resonance.

3. What happens to the reflected field when the cavity gets longer? What about shorter?

If we think about a field Taylor expansion about resonance, to first order we get

$$E_{\text{refl}}(\phi)\big|_{\phi\to 0} = E_{\text{refl}}(0) + \phi\,\frac{dE_{\text{refl}}}{d\phi}(0) = \phi\,\frac{2 i r}{1-r^2}.$$

This means that the field is becomes positive imaginary when the cavity gets longer ($\phi > 0$), but becomes negative imaginary when the cavity gets shorter ($\phi < 0$).
So we’ve finally recovered something which changes linearly with phase.

4. Could you use the reflected field to restore resonance? What would you need in order to do so?

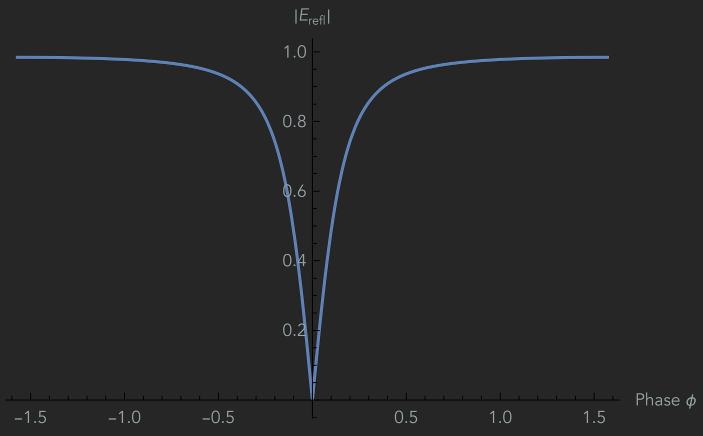
We can’t measure the reflected electric field directly unfortunately.
When we tried to measure the power, we got something that scaled like ϕ^2, which wasn’t that useful.

We would need a *local oscillator* to measure E_{refl} directly.
A local oscillator is just a reference electric field that interferes with our signal field.
We will explore one option for creating a local oscillator below.

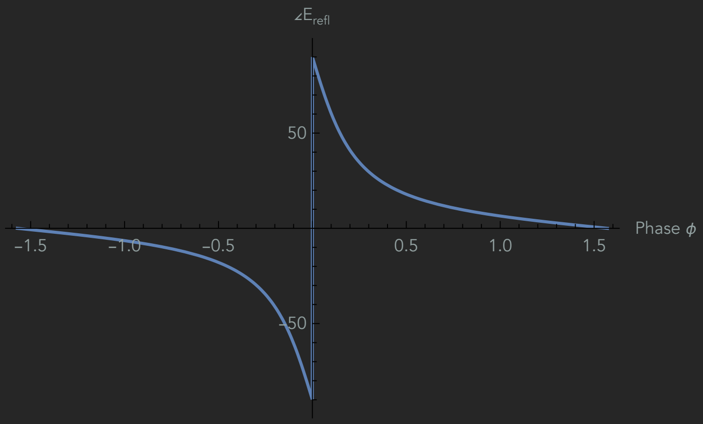
In[303]:=

```
Plot[Abs[Erefl0 //. {r2 → r1, r1 → √(1 - 0.3)}], {ϕ, -π/2, π/2},
 PlotRange → Full, AxesLabel → {"Phase ϕ", "|Erefl|"}]
Plot[180/π Arg[Erefl0 //. {r2 → r1, r1 → √(1 - 0.3)}],
 {ϕ, -π/2, π/2}, PlotRange → Full, AxesLabel → {"Phase ϕ", "∠Erefl"}]
```

Out[303]=



Out[304]=



Part 4. Dither Locking

1. First, find the transfer function from the end mirror E_{cav2} to the reflected field E_{refl}.

$E_{\text{refl}} = r_1 E_{\text{in}} + t_1 e^{-i\phi} E_{\text{cav2}}$, but we are only focused on the contribution to E_{refl} from E_{cav2}, so we set $E_{\text{in}} = 0$

$$\frac{E_{\text{refl}}}{E_{\text{cav2}}} = t_1 e^{-i\phi} \frac{E_{\text{cav2}}}{E_{\text{cav2}}}$$

Remember that $\frac{E_{\text{cav2}}}{E_{\text{cav2}}} = \frac{1}{1-r_1 r_2 \,\text{Exp}[-i\,2\,\phi]}$ is the cavity loop suppression: E_{cav2} has to go around the loop to get back to itself.
This yields our final answer:

$$\frac{E_{\text{refl}}}{E_{\text{cav2}}} = \frac{t_1 e^{-i\phi}}{1-r_1 r_2 e^{-i 2\phi}}$$

In[125]:=

```
                       t1 Exp[-I ϕ]
Ecav2toErefl[ϕ_] := ─────────────────────── ;
                    1 - r1 r2 Exp[-I 2 ϕ]
```

2. Find the carrier field incident on the end mirror.

Incident on the end mirror will be $\frac{E_{\text{cav}}}{E_{\text{in}}} = \frac{t_1}{1-r_1 r_2 e^{-i 2\phi}}$ times the propagation to the back mirror $e^{-i\phi}$.

In[141]:=

```
              t1 Exp[-I ϕ]
Ecavend = ─────────────────────── ;
          1 - r1 r2 Exp[-I 2 ϕ]
```

3. Apply the length modulation Δx cos(ω t) to find the total field immediately reflected off the end mirror E_{cav2}.

Hint: There should be three fields.

We apply the modulation to the E_{cavend} carrier field
$E_{\text{cav2}} = \left(1 + i k\, \Delta x\, e^{i\omega t} + i k\, \Delta x\, e^{-i\omega t}\right) r_2\, E_{\text{cavend}}$

I’ll split the fields explicitly like
$E_{\text{cav2}} = E_{\text{cav2,0}} + E_{\text{cav2,usb}} + E_{\text{cav2,lsb}}$

In[169]:=

```
Ecav20 = r2 Ecavend
Ecav2usb = I k Δx r2 Ecavend Exp[I ω t]
Ecav2lsb = I k Δx r2 Ecavend Exp[-I ω t]
```

Out[169]=

$$\frac{e^{-i\phi}\, r2\, t1}{1 - e^{-2 i \phi}\, r1\, r2}$$

Out[170]=

$$\frac{i\, e^{-i\phi + i t \omega}\, k\, r2\, t1\, \Delta x}{1 - e^{-2 i \phi}\, r1\, r2}$$

Out[171]=

$$\frac{i\, e^{-i\phi - i t \omega}\, k\, r2\, t1\, \Delta x}{1 - e^{-2 i \phi}\, r1\, r2}$$

4. Apply your transfer function from (1) to your E_{cav2} to find the total reflected field.

Hint: It may be easier to consider the accrued phases for carrier ϕ and for the audio

sidebands $\phi \pm \eta$

First, I clarify what the phases ϕ and η are.

$\phi = k L = \frac{\omega_0 L}{c}$ is the phase accrued by the carrier as it passes through the cavity. Here ω_0 is the carrier frequency.

$\eta = \frac{\omega L}{c}$ is the additional phase accrued by the upper sideband (or phase subtracted for the lower sideband) as they pass through the cavity.

The key is, all the beams experience the same cavity length L, but have a different frequency. This causes the fields to accrue a different phase as they pass through the cavity.

So, to find the carrier contribution to E_{refl}, we simply do the usual transfer function $\frac{E_{\text{refl}}}{E_{\text{in}}}$ at the same phase ϕ.
But for the upper and lower sideband contributions to E_{refl}, we need to apply $\frac{E_{\text{refl}}}{E_{\text{cav2}}}$ at $\phi \pm \eta$ to our carrier TF that got us to the end mirror:

Carrier: $E_{\text{refl,0}} = \frac{E_{\text{refl}}}{E_{\text{in}}}(\phi)\, E_{\text{in}}$

Upper Sideband: $E_{\text{refl,usb}} = \frac{E_{\text{refl}}}{E_{\text{cav2}}}(\phi+\eta)\,\frac{E_{\text{cav2}}}{E_{\text{in}}}(\phi)\, E_{\text{in}}$

Lower Sideband: $E_{\text{refl,lsb}} = \frac{E_{\text{refl}}}{E_{\text{cav2}}}(\phi-\eta)\,\frac{E_{\text{cav2}}}{E_{\text{in}}}(\phi)\, E_{\text{in}}$

For the sideband terms, we get a product of the carrier resonance experienced in the cavity, building up to high field strength, and the sideband term itself, which can also resonant inside the cavity

In[172]:=

```
Erefl00 = Erefl0
Ereflusb = Ecav2usb Ecav2toErefl[ϕ + η]
Erefllsb = Ecav2lsb Ecav2toErefl[ϕ + η]

Erefltotal = (Erefl00 + Ereflusb + Erefllsb) E0 Exp[I ω0 t]
```

Out[172]=

$$\frac{r1 - e^{-2 i \phi}\, r2}{1 - e^{-2 i \phi}\, r1\, r2}$$

Out[173]=

$$\frac{i\, e^{-i \phi - i (\eta+\phi) + i t \omega}\, k\, r2\, t1^2\, \Delta x}{\left(1 - e^{-2 i \phi}\, r1\, r2\right) \left(1 - e^{-2 i (\eta+\phi)}\, r1\, r2\right)}$$

Out[174]=

$$\frac{i\, e^{-i \phi - i (\eta+\phi) - i t \omega}\, k\, r2\, t1^2\, \Delta x}{\left(1 - e^{-2 i \phi}\, r1\, r2\right) \left(1 - e^{-2 i (\eta+\phi)}\, r1\, r2\right)}$$

Out[175]=

$$e^{i t \omega 0}\, E0 \left(\frac{r1 - e^{-2 i \phi}\, r2}{1 - e^{-2 i \phi}\, r1\, r2} + \frac{i\, e^{-i \phi - i (\eta+\phi) - i t \omega}\, k\, r2\, t1^2\, \Delta x}{\left(1 - e^{-2 i \phi}\, r1\, r2\right) \left(1 - e^{-2 i (\eta+\phi)}\, r1\, r2\right)} + \frac{i\, e^{-i \phi - i (\eta+\phi) + i t \omega}\, k\, r2\, t1^2\, \Delta x}{\left(1 - e^{-2 i \phi}\, r1\, r2\right) \left(1 - e^{-2 i (\eta+\phi)}\, r1\, r2\right)}\right)$$

5. Calculate the total reflected power $P_{\mathrm{refl}}(t)$. Let $\Delta x^2 \to 0$ for small length modulations.

This is a lot of algebra, and it can be easier to let Mathematica handle it.
Letting Mathematica do the heavy lifting

In[202]:=

```
Prefltotal = Simplify[ComplexExpand[Erefltotal Conjugate[Erefltotal]]];
Prefltotal = FullSimplify[Prefltotal /. {Δx^2 → 0}];
Preflcollect = Collect[Prefltotal, Δx] (*DC term on the Left*)
```

Out[204]=

$$\frac{E0^2 \left(r1^2 + r2^2 - 2\, r1\, r2\, \mathrm{Cos}[2 \phi]\right)}{1 + r1^2\, r2^2 - 2\, r1\, r2\, \mathrm{Cos}[2 \phi]} + \frac{4\, E0^2\, k\, r2\, t1^2\, \Delta x\, \mathrm{Cos}[t \omega] \left(\left(-1 + r1^2\right) r2\, \mathrm{Sin}[\eta] - r1 \left(-1 + r2^2\right) \mathrm{Sin}[\eta + 2 \phi]\right)}{\left(1 + r1^2\, r2^2 - 2\, r1\, r2\, \mathrm{Cos}[2 \phi]\right) \left(1 + r1^2\, r2^2 - 2\, r1\, r2\, \mathrm{Cos}[2 (\eta + \phi)]\right)}$$

However, we can also try to be clever, and develop our intuition,
by leaving the fields in their general form:
(below I have incorporated the imaginary term *i* into the E-fields to be consistent with our Mathematica definitions.)

$$E_{\text{refl}} = E_{\text{refl,0}} + k\,\Delta\text{x}\,E_{\text{refl,usb}}\,e^{i\,\omega\,t} + k\,\Delta\text{x}\,E_{\text{refl,lsb}}\,e^{-i\,\omega\,t}$$

$$P_{\text{refl}}(t) = P_{\text{refl,DC}} + P_{\text{refl,1}\;\omega}(t)$$

$$P_{\text{refl,DC}} = \left|E_{\text{refl,0}}\right|^2 + (k\,\Delta\text{x})^2\left|E_{\text{refl,usb}}\right|^2 + (k\,\Delta\text{x})^2\left|E_{\text{refl,lsb}}\right|^2$$

$$P_{\text{refl,1}\;\omega}(t) = k\,\Delta\text{x}\,P_{\text{in}}\left(e^{i\,\omega\,t}\left(E_{\text{refl,0}}{}^{*}E_{\text{refl,usb}} + E_{\text{refl,0}}\,E_{\text{refl,lsb}}{}^{*}\right) + e^{-i\,\omega\,t}\left(E_{\text{refl,0}}\,E_{\text{refl,usb}}{}^{*} + E_{\text{refl,0}}{}^{*}E_{\text{refl,lsb}}\right)\right)$$

In the last term $P_{\text{refl,1}\;\omega}(t)$, we get our oscillations at frequency ω.

The term on the right inside the parentheses is just the complex conjugate of the term on the left.

So we could do something like write $z(t) = e^{i\,\omega\,t}\left(E_{\text{refl,0}}{}^{*}E_{\text{refl,usb}} + E_{\text{refl,0}}\,E_{\text{refl,lsb}}{}^{*}\right)$, and get

$$P_{\text{refl,1}\;\omega}(t) = k\,\Delta\text{x}\left(e^{i\,\omega\,t}\left(E_{\text{refl,0}}{}^{*}E_{\text{refl,usb}} + E_{\text{refl,0}}\,E_{\text{refl,lsb}}{}^{*}\right) + \text{CC}\right)$$
$$P_{\text{refl,1}\;\omega}(t) = k\,\Delta\text{x}\,(z + z^{*})$$
$$P_{\text{refl,1}\;\omega}(t) = 2\,k\,\Delta\text{x}\,\text{Re}\,(z)$$
$$P_{\text{refl,1}\;\omega}(t) = 2\,k\,\Delta\text{x}\,\text{Re}\left(e^{i\,\omega\,t}\left(E_{\text{refl,0}}{}^{*}E_{\text{refl,usb}} + E_{\text{refl,0}}\,E_{\text{refl,lsb}}{}^{*}\right)\right)$$

If we calculate this term in the parenthesis:

In[179]:=

```
zz =
 Simplify[ComplexExpand[Conjugate[Erefl00] Ereflusb + Erefl00 Conjugate[Erefllsb]]]
```

Out[179]=

$$-\frac{2\,k\,r2\,t1^2\,\Delta x\left(-\left(\left(-1+r1^2\right)r2\,\text{Sin}[\eta]\right)+r1\left(-1+r2^2\right)\text{Sin}[\eta+2\,\phi]\right)\left(\text{Cos}[t\,\omega]+i\,\text{Sin}[t\,\omega]\right)}{\left(1+r1^2\,r2^2-2\,r1\,r2\,\text{Cos}[2\,\phi]\right)\left(1+r1^2\,r2^2-2\,r1\,r2\,\text{Cos}[2\,(\eta+\phi)]\right)}$$

In[188]:=

```
Simplify[ComplexExpand[Re[zz]]]
```

Out[188]=

$$-\frac{2\,k\,r2\,t1^2\,\Delta x\,\text{Cos}[t\,\omega]\left(-\left(\left(-1+r1^2\right)r2\,\text{Sin}[\eta]\right)+r1\left(-1+r2^2\right)\text{Sin}[\eta+2\,\phi]\right)}{\left(1+r1^2\,r2^2-2\,r1\,r2\,\text{Cos}[2\,\phi]\right)\left(1+r1^2\,r2^2-2\,r1\,r2\,\text{Cos}[2\,(\eta+\phi)]\right)}$$

In[193]:=

```
Prefl1ω = Simplify[2 E0^2 Simplify[ComplexExpand[Re[zz]]]]
```

Out[193]=

$$-\frac{4\,E0^2\,k\,r2\,t1^2\,\Delta x\,\text{Cos}[t\,\omega]\left(-\left(\left(-1+r1^2\right)r2\,\text{Sin}[\eta]\right)+r1\left(-1+r2^2\right)\text{Sin}[\eta+2\,\phi]\right)}{\left(1+r1^2\,r2^2-2\,r1\,r2\,\text{Cos}[2\,\phi]\right)\left(1+r1^2\,r2^2-2\,r1\,r2\,\text{Cos}[2\,(\eta+\phi)]\right)}$$

Compare Prefl1 ω to the second term in Prefltotal, and we get the same result

In[205]:=

```
Preflcollect⟦2⟧
```

Out[205]=

$$\frac{4\,E0^2\,k\,r2\,t1^2\,\Delta x\,\text{Cos}[t\,\omega]\left(\left(-1+r1^2\right)r2\,\text{Sin}[\eta]-r1\left(-1+r2^2\right)\text{Sin}[\eta+2\,\phi]\right)}{\left(1+r1^2\,r2^2-2\,r1\,r2\,\text{Cos}[2\,\phi]\right)\left(1+r1^2\,r2^2-2\,r1\,r2\,\text{Cos}[2\,(\eta+\phi)]\right)}$$

In[207]:=

```
Simplify[Preflcollect[[2]] / Prefl1ω]
```

Out[207]=

```
1
```

6. Demodulate the power signal

This appears to be another monumental algebraic task, and it is.
Again, let’s start by letting Mathematica do the heavy lifting.

In[243]:=

```
Prefltotaldemod I =
 Simplify[1/(2 π) Integrate[Prefltotal Cos[ωt] /. {t → ωt/ω}, {ωt, 0, 2 π}]]
PrefltotaldemodQ =
 Simplify[1/(2 π) Integrate[Prefltotal Sin[ωt] /. {t → ωt/ω}, {ωt, 0, 2 π}]]
Prefltotaldemodtf = (PrefltotaldemodI + I PrefltotaldemodQ) / Δx
```

Out[243]=

$$\frac{2\,\mathrm{E0}^2\,k\,\mathrm{r2}\,\mathrm{t1}^2\,\Delta x\left(\left(-1+\mathrm{r1}^2\right)\mathrm{r2}\,\mathrm{Sin}[\eta]-\mathrm{r1}\left(-1+\mathrm{r2}^2\right)\mathrm{Sin}[\eta+2\,\phi]\right)}{\left(1+\mathrm{r1}^2\,\mathrm{r2}^2-2\,\mathrm{r1}\,\mathrm{r2}\,\mathrm{Cos}[2\,\phi]\right)\left(1+\mathrm{r1}^2\,\mathrm{r2}^2-2\,\mathrm{r1}\,\mathrm{r2}\,\mathrm{Cos}[2\,(\eta+\phi)]\right)}$$

Out[244]=

```
0
```

Out[245]=

$$\frac{2\,\mathrm{E0}^2\,k\,\mathrm{r2}\,\mathrm{t1}^2\left(\left(-1+\mathrm{r1}^2\right)\mathrm{r2}\,\mathrm{Sin}[\eta]-\mathrm{r1}\left(-1+\mathrm{r2}^2\right)\mathrm{Sin}[\eta+2\,\phi]\right)}{\left(1+\mathrm{r1}^2\,\mathrm{r2}^2-2\,\mathrm{r1}\,\mathrm{r2}\,\mathrm{Cos}[2\,\phi]\right)\left(1+\mathrm{r1}^2\,\mathrm{r2}^2-2\,\mathrm{r1}\,\mathrm{r2}\,\mathrm{Cos}[2\,(\eta+\phi)]\right)}$$

So we retrieve half the 1 ω power signal in I, and nothing in the Q quadrature.
Forming the length to reflected power transfer function at ω is as simple as dividing by Δx, as we did in the final step.

This can be seen by analyzing Prefl1 ω above.
It has only a Cos[ω t] term, which indicates which quadrature we’ll find our signal in (the I-quadrature).

Again, by being a bit clever we can simplify our calculations.
Starting again with the generalized term:

$$P_{\mathrm{refl},1\ \omega}(t) = 2\,k\,\Delta\mathrm{x}\,\mathrm{Re}\,(z)$$

$$\text{with } z = e^{i\,\omega\,t}\left(E_{\mathrm{refl},0}{}^{*}\,E_{\mathrm{refl,usb}} + E_{\mathrm{refl},0}\,E_{\mathrm{refl,lsb}}{}^{*}\right)$$

We can try to directly demodulate this expression.

$$P_{\mathrm{refl},I} = \frac{1}{2\,\pi}\int_0^{2\,\pi} P_{\mathrm{refl},1\ \omega}(t)\cos(\omega\mathrm{t})\,d(\omega\mathrm{t})$$

$$P_{\text{refl},Q} = \frac{1}{2\pi}\int_0^{2\pi} P_{\text{refl},1\ \omega}(t)\sin(\omega t)\,d(\omega t)$$

If I set z = $y\,e^{i\omega t}$, where $y = E_{\text{refl},0}{}^{*} E_{\text{refl,usb}} + E_{\text{refl},0}\ E_{\text{refl,lsb}}{}^{*}$,

then it becomes exquisitely clear what we are measuring when we do the demodulation

```
In[232]:=
      1
II = ─── Integrate[ComplexExpand[Re[y Exp[I ωt]] Cos[ωt], y], {ωt, 0, 2 π}]
     2 π
      1
QQ = ─── Integrate[ComplexExpand[Re[y Exp[I ωt]] Sin[ωt], y], {ωt, 0, 2 π}]
     2 π
II + I QQ
TF = FullSimplify[II + I QQ]
```

```
Out[232]=
 Re[y]
 ─────
   2

Out[233]=
   Im[y]
 - ─────
     2

Out[234]=
   1              Re[y]
 - ─ ⅈ Im[y] + ─────
   2                2

Out[235]=
 Conjugate[y]
 ────────────
      2
```

By using modulating the end mirror with a length change Δx cos(ω t),
and then demodulating at that same frequency ω,
we are picking off a product of our electric fields $y = E_{\text{refl},0}{}^{*} E_{\text{refl,usb}} + E_{\text{refl},0}\ E_{\text{refl,lsb}}{}^{*}$.

What’s more, that product of electric fields is incredibly useful for our ultimate goal: maintaining resonance by detecting the phase of light via a power signal.
Essentially, encoded into the power signal $P_{\text{refl}}(t)$ at the frequency ω is our phase signal from E_{refl}, which can be used to hold onto resonance.

Plot of Dither Locking Phase Sweep

Let’s plot a phase sweep of our demodulated cavity length to reflected power transfer function
$\frac{P_{\text{refl}}}{\Delta x}(\omega, \phi)$
for some low frequency signal, say $\omega = 2\pi$ (100 Hz) for some cavity length L = 1 meter.
We’ll assume some moderate finesse cavity, with equal transmissions $T_1 = T_2 = 0.3$.

Parameters for substitution into our function

In[293]:=

```
params =
  {E0 → 1, k → 2π/λ, ω → 2 π f, η → ω L/c, r1 → √(1 - T1), r2 → √(1 - T2), t1 → √T1, t2 → √T2};
values = {λ → 1064 × 10^-9, T1 → 0.3, T2 → 0.3, L → 1, c → 3 × 10^8, f → 100};
```

In[300]:=

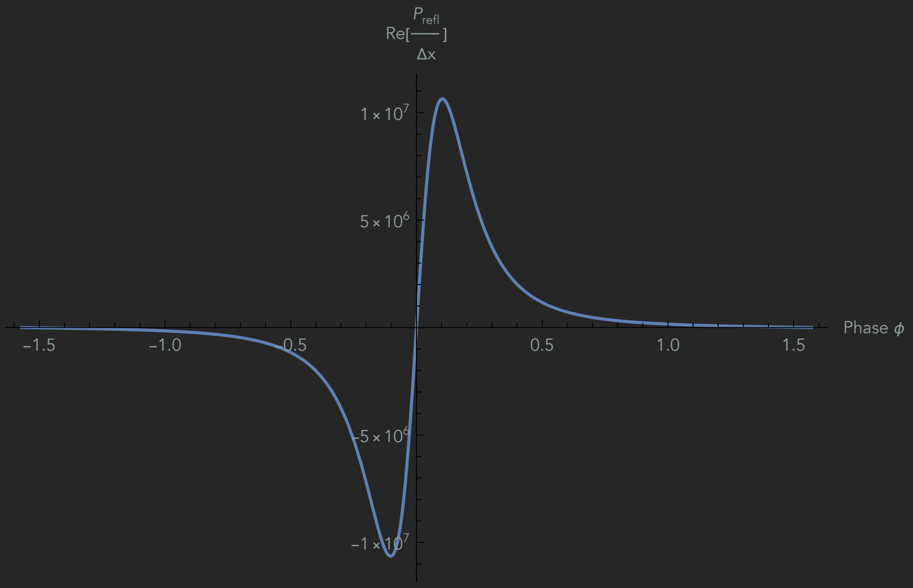
```
(*Prefltotaldemodtf//.params/.values*)
```

Plot of our demodulated function $\frac{P_{\text{refl}}}{\Delta x}(\omega, \phi)$ over ϕ

In[311]:=

```
Plot[Re[Prefltotaldemodtf //. params /. values], {ϕ, -π/2, π/2},
 PlotRange → Full, AxesLabel → {"Phase ϕ", "Re[Prefl/Δx]"}]
(*Plot[Im[Prefltotaldemodtf//.params/.values],
 {ϕ,-π/2,π/2},PlotRange→Full,AxesLabel→{"Phase ϕ","Im[Prefl/Δx]"}]*)
```

Out[311]=



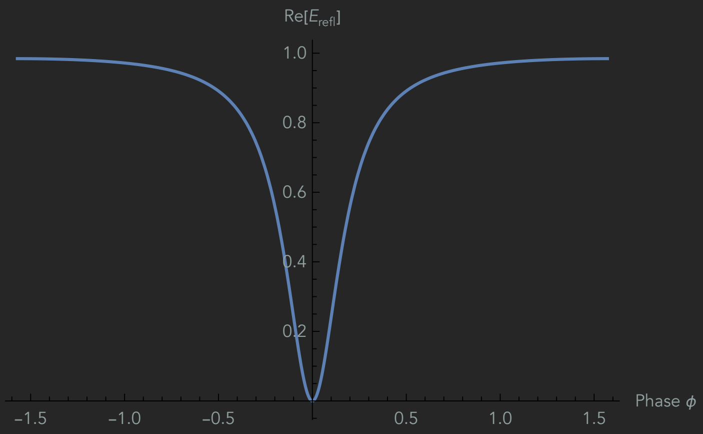
Interpreting the above plot, we get a zero crossing of our reflected power signal P_{refl} coming from our end-mirror modulation Δx directly at resonance where $\phi = 0$.
The y-axis units are in watts per meter, and tell us ultra-sensitive our interferometer is to end-mirror motion near resonance, even for this low-performing interferometer..

Comparing to the real and imaginary plots of E_{refl} sweep over ϕ.
Notice that our real demodulated signal $\frac{P_{\text{refl}}}{\Delta x}(\omega, \phi)$ is very similar to our imaginary part of E_{refl}

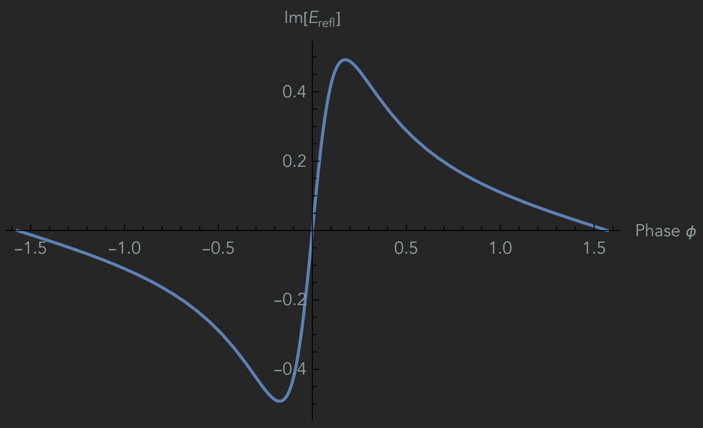
In[322]:=

```
Plot[Re[Erefl0 //. {r2 → r1, r1 → √(1 - 0.3)}], {ϕ, -π/2, π/2},
 PlotRange → Full, AxesLabel → {"Phase ϕ", "Re[Erefl]"}]
Plot[Im[Erefl0 //. {r2 → r1, r1 → √(1 - 0.3)}], {ϕ, -π/2, π/2},
 PlotRange → Full, AxesLabel → {"Phase ϕ", "Im[Erefl]"}]
```

Out[322]=



Out[323]=



Plot of Dither Locking Transfer Function

Here we adjust the dither frequency ω, while assuming we are perfectly on resonance.
This gives us our *length-to-reflected-power transfer function* frequency response.

This time our values need to specify our carrier phase ϕ, not our modulation frequency ω which we want to sweep.
We specify the carrier phase $\phi = 1^{\circ}$, because if we are at exactly $\phi = 0^{\circ}$ our error signal goes to zero (as seen in our phase sweep above).

```
In[360]:=
values2 = {λ → 1064 × 10^-9, T1 → 0.3, T2 → 0.3, L → 1, c → 3 × 10^8, ϕ → 1 π/180};
flow = 10^4;
fhigh = 3 × 10^8;
```

```
In[342]:=
Prefltotaldemodtf //. params
```

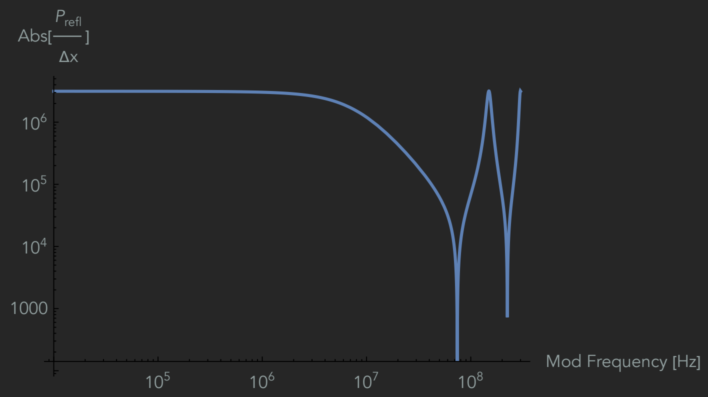
Out[342]=

$$\left(4\,\pi\,\mathrm{T1}\,\sqrt{1-\mathrm{T2}}\,\left(-\mathrm{T1}\,\sqrt{1-\mathrm{T2}}\,\mathrm{Sin}\left[\frac{2\,f\,L\,\pi}{c}\right]+\sqrt{1-\mathrm{T1}}\,\mathrm{T2}\,\mathrm{Sin}\left[\frac{2\,f\,L\,\pi}{c}+2\,\phi\right]\right)\right)\Big/\left(\lambda\,\left(1+(1-\mathrm{T1})\,(1-\mathrm{T2})-2\,\sqrt{1-\mathrm{T1}}\,\sqrt{1-\mathrm{T2}}\,\mathrm{Cos}[2\,\phi]\right)\left(1+(1-\mathrm{T1})\,(1-\mathrm{T2})-2\,\sqrt{1-\mathrm{T1}}\,\sqrt{1-\mathrm{T2}}\,\mathrm{Cos}\left[2\,\left(\frac{2\,f\,L\,\pi}{c}+\phi\right)\right]\right)\right)$$

In[363]:=

```
LogLogPlot[Abs[Prefltotaldemodtf //. params /. values2], {f, flow, fhigh},
 PlotRange → Full, AxesLabel → {"Mod Frequency [Hz]", "Abs[Prefl/Δx]"}]
LogLinearPlot[180/π Arg[Prefltotaldemodtf //. params /. values2], {f, flow, fhigh},
 PlotRange → Full, AxesLabel → {"Mod Frequency [Hz]", "Arg[Prefl/Δx]"}]
```

Out[363]=



Out[364]=